# Representations

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In the first extra credit assignment, we took a matrix given by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and found two eigenvalue-eigenvector pairs,

$$\lambda_1 = i \qquad v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -i \qquad v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

The first thing to notice about $A$ is that if $\vec{x}$ is a vector in $\mathbb{R}^2$ (that is, if $\vec{x}$ is a point on the plane), then $A\vec{x}$ is simply $\vec{x}$ rotated 90 degrees counterclockwise about the origin. For instance, suppose that

$$\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This vector (or point) lives $\sqrt{2}$ units away from the origin, in the first quadrant, on the line $y = x$. If we rotated it 90 degrees counterclockwise about the origin, it would end up at the point $(-1, 1)$, which is also $\sqrt{2}$ units from the origin, in the second quadrant, on the line $y = -x$. (Draw these points right now!) And it's *really* easy to check that

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

If you're still not convinced that $A$ rotates each point on the $\mathbb{R}^2$ plane 90 degrees counterclockwise, I encourage you to pick some of your favorite points on the plane, $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$, find where a 90 degree rotation *should* take them, and then compute $A\vec{x}_1, A\vec{x}_2, \ldots, A\vec{x}_n$. You'll see that $A$ rotates each of your favorite points 90 degrees counterclockwise about the origin. Of course, you could always write a formal proof ...

We'll keep all of this in the back of our minds for now. Let's talk about how we represent points in the plane. Thus far, we've thought of each point as a column vector in $\mathbb{R}^2$. Given any point $(a, b)$ in the plane, we construct a column vector $\vec{v} \in \mathbb{R}^2$, by making the $x$-coordinate the first vector entry and the $y$-coordinate the second entry. That is,

$$(a, b) \to \begin{pmatrix} a \\ b \end{pmatrix}.$$

Let's define a new way to represent a point in the plane. We'll say that if $(a, b)$ is a point in the plane, then we will represent it like this,

$$(a, b) \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

The container for $(a, b)$ may look different, but the exact same information is encoded in both

$$\begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Also, if we're given a matrix of the form

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix},$$

we know that this matrix represents the point $(x, y)$.

Note that two column vectors cannot be multiplied together to give another column vector, but two $2 \times 2$ matrices can be multiplied to give a $2 \times 2$ matrix. Our new representation gives us a way to multiply two points in the plane. Suppose that $(a, b)$ and $(c, d)$ are two points in the plane. Then we can represent these points as the matrices

$$Z_1 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{and} \quad Z_2 = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}.$$

Let $Z_3 = Z_1 Z_2$. Then $Z_3$ is given by

$$\begin{aligned} Z_3 &= Z_1 Z_2 \\ &= \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \\ &= \begin{pmatrix} ac - bd & -ad - bc \\ bc + ad & -bd + ac \end{pmatrix} \\ &= \begin{pmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{pmatrix} \\ &= \begin{pmatrix} x_3 & -y_3 \\ y_3 & x_3 \end{pmatrix} \end{aligned}$$

Since $Z_3$ has the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, then $Z_3$ corresponds to the point $(x_3, y_3) = (ac - bd, ad + bc)$. But wait a minute. This looks really familiar. We took the points $(a, b)$ and $(c, d)$, multiplied them together and got the point $(ac - bd, ad + bc)$. That's exactly what we'd get by multiplying the two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ !! Check it out. Let $z_3 = z_1 z_2$. Then,

$$\begin{aligned} z_3 &= z_1 z_2 \\ &= (a + ib)(c + id) \\ &= ac + iad + ibc + i^2 bd \\ &= ac + iad + ibc - bd \\ &= ac - bd + iad + ibc \\ &= (ac - bd) + i(ad + bc). \end{aligned}$$

Since we've defined our representation by

$$(a, b) \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

and the complex-valued point $z_3$ has $x$-coordinate $(ac - bd)$ and $y$-coordinate $(ad + bc)$, we see that

$$z_3 = (ac - bd, ad + bc) \to \begin{pmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{pmatrix},$$

so that $z_3$ and $Z_3$ represent the same point. In this way, multiplication of two complex numbers can be represented as the multiplication of two real-valued matrices in $\mathbb{R}^{2 \times 2}$. Let's return to our matrix $A$ from the extra credit assignment.

Since $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, then $A$ can be represented as a complex number, $A = x+iy$. Substituting the values from the matrix, we find that $A = i$. And indeed, if you multiply any point $a+ib$ in the plane by $i$, you get a 90 degree rotation counterclockwise about the origin since

$$(a+ib)\,i = ia + i^2 b = -b + ia.$$

Pretty cool, huh?